MODULAR FORMS 2019: DIRICHLET SERIES AND EULER PRODUCTS WEEK OF MAY 26, 2019

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Contents

0.1. **Arithmetic functions and Dirichlet series.** An arithmetic function is simply a function $a:\mathbb{N}\to\mathbb{C}$ on the natural numbers. Examples are the constant function $\mathbf{1}$, the power functions n^α.

The Dirichlet convolution of two arithmetic functions $a,b:\mathbb{N}\to\mathbb{C}$ is defined by

$$a*b(n) := \sum_{d\mid n} a(d)b(\frac{n}{d})$$

For instance, if we denote by $\mathbf{1}$ the constant function 1, and

$$\sigma_0(n) = \#\{d\geq 1, d\mid n\} = \sum_{d\mid n} 1$$

the number of divisors of n, then clearly

$$\mathbf{1}*\mathbf{1} = \sigma_0$$

Likewise, the divisor sums

$$\sigma_\alpha(n) = \sum_{d\mid n} d^\alpha$$

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are clearly the convolution

$$\sigma_\alpha = \mathbf{1} * n^\alpha$$

To any arithmetic function $a : \mathbb{N} \to \mathbb{C}$, which is of polynomial growth: $|a(n)| \ll n^A$, we associate a Dirichlet series

$$D_a(s) := \sum_{n\geq 1} a(n)n^{-s}, \quad \mathrm{Re}(s) > A+1$$

which converges for $\mathrm{Re}(s) > A$, and uniformly in any closed half-plane $\mathrm{Re}(s) \geq A+\delta$, hence defines a holomorphic function for $\mathrm{Re}(s) > A$.

For instance, for the constant function $\mathbf{1}$ we have

$$D_{\mathbf{1}}(s) = \sum_{n\geq 1} \frac{1}{n^s} = \zeta(s)$$

is the Riemann zeta function.

For the power function n^α, we have

$$\sum_{n\geq 1} \frac{n^\alpha}{n^s} = \zeta(s-\alpha), \quad \mathrm{Re}(s) > 1 + \mathrm{Re}(\alpha)$$

Lemma 0.1. *The Dirichlet series attached to a convolution is the product of the two Dirichlet series:*

$$D_{a*b}(s) = D_a(s)D_b(s)$$

Corollary 0.2. *The Dirichlet series $D_\alpha(s) = \sum_{n\geq 1} \sigma_\alpha(n)n^{-s}$ associated to the divisors sum σ_α is the product*

$$D_\alpha(s) = \zeta(s)\zeta(s-\alpha)$$

An arithmetic function is multiplicative if $a(1) \neq 0$ and $a(mn) = a(m)a(n)$ whenever m, n are coprime. Necessarily then $a(1) = 1$.

Examples: the constant function $\mathbf{1}$, the power functions n^α are multiplicative (in fact completely multiplicative).

Lemma 0.3. *If $a, b : \mathbb{N} \to \mathbb{C}$ are multiplicative then so is their convolution $a * b$.*

Since $\sigma_\alpha = \mathbf{1} * n^\alpha$, we obtain

Lemma 0.4. *The divisor sums σ_α are multiplicative: $\sigma_\alpha(mn) = \sigma_\alpha(m)\sigma_\alpha(n)$ if $\gcd(m,n) = 1$.*

Exercise 1. The divisor sums σ_α satisfy for p prime and $r \geq 1$

$$\sigma_\alpha(p)\sigma_\alpha(p^r) = \sigma_\alpha(p^{r+1}) + p^\alpha \sigma_\alpha(p^{r-1})$$

Lemma 0.5. *If $a : \mathbb{N} \to \mathbb{C}$ is multiplicative then for $\mathrm{Re}(s) \gg 1$,*

$$D_a(s) = \prod_{p \text{ prime}} \sum_{r\geq 0} a(p^r)p^{-rs}$$

0.2. **The Dirichlet series attached to a modular form.** Let $0 \neq f \in M_k$ be a modular form of weight k, with Fourier expansion $f = \sum_{n\geq 0} a_f(n)q^n$. The associated Dirichlet series is, for $s \in \mathbb{C}$, $\mathrm{Re}(s) \gg 1$

$$D(s, f) := \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}$$

Note that we ignore the zero'th coefficient $a_f(0)$. Recall that we showed that for any $F \in M_k$, the Fourier coefficients satisfy $|a_f(n)| \ll n^{k-1}$ (and better bounds for cusp forms). Hence the series converges for $\mathrm{Re}(s) > k$, and uniformly in any closed half-plane $\mathrm{Re}(s) \geq k+\delta$, hence defines a holomorphic function for $\mathrm{Re}(s) \gg 1$.

0.3. **Eisenstein series.** The normalized Eisenstein series has Fourier expansion

$$E_k = 1 + \gamma_k \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n$$

Hence the associated Dirichlet series is (a multiple of)

$$D(s) = \sum_{n\geq 1} \sigma_{k-1}(n)q^n$$

We have already computed this Dirichlet series, so that we find

$$D(s, E_k) = \zeta(s)\zeta(s-(k-1))$$

0.4. **Analytic continuation and functional equation.** The Gamma function is defined for $\mathrm{Re}(s) > 0$ as

$$\Gamma(s) := \int_0^{\infty} e^{-t}t^s \frac{dt}{t}$$

Integration by parts shows that in this regime, we have the functional equation

$$\Gamma(s+1) = s\Gamma(s)$$

so that in particular we obtain that for $n \geq 1$ integer, $\Gamma(n+1) = n!$ and using the functional equation we obtain that Γ has meromorphic continuation to the entire complex plane, save for simple poles at the non-negative integers $s = 0, -1, -2, -3, \ldots$.

Exercise 2. Compute the residue $\mathrm{Res}_{s=-n}\Gamma(s)$.

Theorem 0.6. *Let $f \in S_k$ be a cusp form ($k \geq 12$ even). Set $\Lambda_f(s) := (2\pi)^{-s}\Gamma(s)D(s,f)$, initially defined for $\mathrm{Re}(s) \gg 1$. Then $D(s,f)$ admits an analytic continuation to the entire complex plane, and satisfies the functional equation*

$$\Lambda_f(s) = i^{-k}\Lambda_f(k-s)$$

Proof. We first give an integral representation of $\Lambda_f(s)$. Consider the integral (Mellin transform)

$$I(s) = \int_0^\infty f(iy) y^s \frac{dy}{y}$$

Recall that a cusp form decays exponentially at infinity: $|f(x+iy)| \ll e^{-2\pi y}$ as $y \to +\infty$, so the integral converges at $y = \infty$ for all s, and for $y \to 0$, we use the modular transformation formula $f(-1/\tau) = \tau^k f(\tau)$ for $\tau = iy$

$$f(iy) = (\frac{1}{iy})^k f(-\frac{1}{iy})$$

to deduce that $|f(iy)| \ll y^{-k} e^{-2\pi/y}$ as $y \to 0$, so that the integral also converges at $y = 0$ if for all s. Hence $I(s)$ is an entire function.

Now insert the Fourier expansion

$$f(iy) = \sum_{n\geq 1} a_f(n) e^{-2\pi n y}$$

to find

$$I(s) = \sum_{n\geq 1} a_f(n) \int_0^\infty e^{-2\pi n y} y^s \frac{dy}{y}$$

Changing variables gives

$$\int_0^\infty e^{-2\pi n y} y^s \frac{dy}{y} = (2\pi n)^{-s} \Gamma(s)$$

so that

$$I(s) = (2\pi)^{-s} \Gamma(s) \sum_{n\geq 1} a_f(n) n^{-s} =: \Lambda_f(s)$$

which shows that $\Lambda_f(s)$ is entire.

It remains to prove the functional equation. Separate the integral as

$$I(s) = \int_{y=0}^1 + \int_1^\infty$$

Using the transformation formula for f, we write

$$\int_0^1 f(iy) y^s \frac{dy}{y} = \int_0^1 (\frac{1}{iy})^k f(-\frac{1}{iy}) y^s \frac{dy}{y} = i^{-k} \int_0^1 f(\frac{i}{y}) y^{s-k} \frac{dy}{y}$$

Now change variables $y' = 1/y$:

$$= i^{-k} \int_1^\infty f(iy')(y')^{k-s} \frac{dy'}{y'}$$

so that we obtain

$$I(s) = \int_1^\infty f(iy) \Big(y^s + i^{-k} y^{k-s} \Big) \frac{dy}{y}$$

Hence, using $i^{-k} = i^k$ for k even,

$$I(k-s) = \int_1^\infty f(iy)\Big(y^{k-s} + i^{-k}y^s\Big)\frac{dy}{y} = i^k I(s)$$

□

0.5. **Analytic continuation and functional equation for Riemann's zeta function.** The above proof of the functional equation for Dirichlet series attached to cusp forms is modeled on one of Riemann's proofs of the corresponding fact for the Riemann zeta function, except that there is an extra step which leads to a pole. The result is

Theorem 0.7. *Let $\zeta^*(s) := \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$. Then $\zeta^*(s)$ is analytic except for simple poles at $s = 0$ and $s = 1$, and has a functional equation*

$$\zeta^*(s) = \zeta^*(1-s)$$

Proof. The completed Riemann zeta function ζ^* is essentially the Mellin transform of the one variable theta function

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2} = \sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau}$$

Precisely, set

$$\omega(y) = \frac{1}{2}\Big(\theta(iy) - 1\Big) = \sum_{n\geq 1} e^{-\pi n^2 y}$$

and consider the integral

$$I(s) := \int_0^\infty \omega(y) y^{s/2}\frac{dy}{y}$$

The integral converges for all s at $y = \infty$, since $\omega(y) \ll e^{-\pi y}$ as $y \to \infty$.

To understand convergence at $y = 0$, recall the transformation formula of the one-variable theta function

$$\theta(-1/\tau) = \sqrt{-i\tau}\,\theta(\tau)$$

In particular, taking $\tau = iy$ (which is the way we proved it...)

$$\theta(\frac{i}{y}) = \sqrt{y}\,\theta(iy)$$

so that as $y \to 0$, since $\theta(iy) \to 1$ as $y \to 0$,

$$\theta(iy) \sim \frac{1}{\sqrt{y}}$$

and hence

$$\omega(\frac{1}{y}) \sim \frac{1}{2\sqrt{y}}$$

Thus the integral $I(s)$ converges at $y = 0$ like that of $\int_0^1 \frac{1}{2\sqrt{y}} y^{s/2} \frac{dy}{y}$, namely for $\mathrm{Re}(s) > 1$.

Then for $\mathrm{Re}(s) > 1$,

$$I(s) = \sum_{n\geq 1} \int_0^\infty e^{-\pi n^2 y} y^{s/2} \frac{dy}{y} = \sum_{n\geq 1} (\pi n^2)^{-s/2} \Gamma(\frac{s}{2}) =: \zeta^*(s)$$

Now split the integral as

$$I(s) = \int_0^1 + \int_1^\infty$$

The integral $\int_1^\infty$ is absolutely convergent for all $s \in \mathbb{C}$ so is an entire function. To treat the integral $\int_0^1$, change variables $y = 1/t$

$$\int_0^1 \omega(y) y^{s/2} \frac{dy}{y} = \int_1^\infty \omega(\frac{1}{t}) t^{-s/2} \frac{dt}{t}$$

The functional equation of theta $\theta(i/t) = \sqrt{t}\theta(it)$ gives

$$\omega(\frac{1}{t}) = \frac{1}{2}\Big(\sqrt{t}\theta(it) - 1\Big) = \sqrt{t}\omega(t) + \frac{\sqrt{t}-1}{2}$$

Hence

$$\int_1^\infty \omega(\frac{1}{t}) t^{-s/2} \frac{dt}{t} = \int_1^\infty \omega(t) t^{\frac{1-s}{2}} \frac{dt}{t} + \int_1^\infty \frac{\sqrt{t}-1}{2} t^{-s/2} \frac{dt}{t}$$

The first integral converges for all s, hence is an entire function of s, while the second integral is explicitly evaluated as

$$\int_1^\infty \frac{\sqrt{t}-1}{2} t^{-s/2} \frac{dt}{t} = -\frac{1}{1-s} - \frac{1}{s}$$

Thus

$$\zeta^*(s) = -\frac{1}{1-s} - \frac{1}{s} + \int_1^\infty \omega(t) \Big(t^{\frac{s}{2}} + t^{\frac{1-s}{2}}\Big) \frac{dt}{t}$$

Both summands are clearly symmetric under $s \mapsto 1-s$, so that $\zeta^*(s) = \zeta^*(1-s)$, and the integral is entire, and so we find that $\zeta^*(s)$ has an analytic continuation to all of $\mathbb{C}$ except for simple poles at $s = 0, 1$. □

Corollary 0.8. *The Riemann zeta function has an analytic continuation to the entire complex plane except for a simple pole at $s = 1$, where* $\mathrm{Res}_{s=1} \zeta = 1$*,*

This is because $\Gamma \neq 0$ and so ζ does not have any more poles than ζ^*.

Corollary 0.9. $\zeta(-2n) = 0$ *for* $n = 1, 2, \dots$.

These are called the "trivial" zeros of $\zeta(s)$. The nontrivial zeros are the zeros of ζ^*.

0.6. **Euler products for Hecke eigenforms.** Assume that $f \in S_k$ is a normalized Hecke eigenform: $T(n)f = \lambda_f(n)f$, $a_f(1) = 1$, so that the Fourier expansion is

$$f(\tau) = \sum_{n \geq 1} \lambda_f(n) q^n$$

The corresponding Dirichlet series is then

$$D(s, f) = \sum_{n \geq 1} \lambda_f(n) n^{-s}$$

Since λ_f is multiplicative, we have

$$D(s, f) = \prod_{\text{prime}} \sum_{r \geq 0} \lambda_f(p^r) p^{-rs}$$

Lemma 0.10.

$$\sum_{r=0}^{\infty} \lambda_f(p^r) X^r = \frac{1}{1 - \lambda_f(p) X + p^{k-1} X^2}$$

This is equivalent to the recursion

$$\lambda_f(p)\lambda_f(p^r) = \lambda_f(p^{r+1}) + p^{k-1}\lambda_f(p^{r-1}), \quad r \geq 1$$

Corollary 0.11. *Let $f \in S_k$ be a cuspidal Hecke eigenform. Then*

$$D(s, f) = \prod_p (1 - \lambda_f(p) p^{-s} + p^{k-1} p^{-2s})^{-1}$$

0.7. **The Riemann Hypothesis for $L(s, f)$.** If $f \in S_k$ is a Hecke eigenform, then we saw that the corresponding Dirichlet series admits an Euler product

$$D(s, f) = \sum_{n \geq 1} \lambda_f(n) n^{-s} = \prod_p (1 - \lambda_f(p) p^{-s} + p^{k-1-2s})^{-1}$$

Writing the p-the factor as

$$\begin{aligned} 1 - \lambda_f(p) X + p^{k-1} X^2 &= (1 - \alpha_1(p) p^{(k-1)/2} X)((1 - \alpha_2(p) p^{(k-1)/2} X) \\ &= \det(I - X p^{(k-1)/2} A_p) \end{aligned}$$

where

$$A_p := \begin{pmatrix} \alpha_1(p) & 0 \\ 0 & \alpha_2(p) \end{pmatrix}, \quad p^{k-1} \operatorname{tr} A_p = \lambda_f(p), \quad \det A_p = 1$$

we have

$$D(s,f) = \prod_p \det(I - p^{-s}p^{(k-1)/2}A_p)^{-1}$$

Deligne's theorem (Ramanujan's conjecture) $|\lambda_f(p)| \leq 2p^{(k-1)/2}$ is equivalent to $A_p \in \mathrm{SU}(2)$ is unitary.

Lets normalize differently: set

$$L(s,f) := D(s + \frac{k-1}{2}, f) = \prod_p \det(I - p^{-s}A_p)^{-1}$$

and

$$L^*(s,f) = (2\pi)^{-(s+\frac{k-1}{2})}\Gamma(s + \frac{k-1}{2})D(s + \frac{k-1}{2}, f)$$

which now satisfies a functional equation

$$L^*(s,f) = i^k L^*(1-s,f)$$

whose symmetry axis is the line $\mathrm{Re}(s) = \frac{1}{2}$. The analogue of the Riemann Hypothesis is that all zeros of $L^*(s,f)$ (which are called the non-trivial zeros of $L(s,f)$) lie on the line $\mathrm{Re}(s) = 1/2$.

This has not been established in any example.

0.8. **The converse theorem.** We saw that a modular form gives a Dirichlet series with analytic continuation and a certain specific functional equation. It turns out that Dirichlet series with this precise functional equation must correspond to modular forms. This is Hecke's "converse theorem" (1936) for $\mathrm{SL}(2,\mathbb{Z})$.

Theorem 0.12. *Let* $D(s) = \sum_{n\geq 1} a(n)n^{-s}$ *be a Dirichlet series, with* $|a(n)| \ll n^{\nu}$ *for some* $\nu > 0$ *(so is absolutely convergent in* $\mathrm{Re}(s) \gg 1$*), so that*

(1) $D(s)$ *admits an analytic continuation to all of* $\mathbb{C}$
(2) $D(s)$ *satisfies the functional equation (*$k \geq 12$ *even)*

$$\Lambda(s) := (2\pi)^{-s}\Gamma(s)D(s) = (-1)^{k/2}D(k-s),$$

(3) $D(s)$ *is bounded in vertical strips: Given* $-\infty < \alpha < \beta < +\infty$*, there is some* $C(\alpha,\beta)$ *so that* $|D(\sigma + it)| < C(\alpha,\beta)$ *for* $\sigma \in [\alpha,\beta]$.

Then there is some $f \in S_k$ *so that* $D(s) = D(s,f)$.

The proof is a simple application of Mellin inversion. One forms the function $f(\tau) := \sum_{n\geq 1} a(n)q^n$, which by definition satisfies $f(\tau+1) = f(\tau)$ and is holomorphic in $|q| < 1$ (i.e. $\tau \in \mathbb{H}$) since $|a(n)| \ll n^{\nu}$ and vanishes at $q = 0$, so all that is left is to establish the transformation formula $f(-1/\tau) = \tau^k f(\tau)$. Since both sides are analytic in τ, it suffices to do so for $\tau = iy$, $y > 0$. This is done by using Mellin inversion

$$e^{-t} = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=2} \Gamma(s) t^{-s} ds$$

so that

$$\begin{aligned} f(iy) = \sum_{n \geq 1} a(n) e^{-2\pi n y} &= \sum_{n \geq 1} a(n) \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=2} \Gamma(s)(2\pi n y)^{-s} ds \\ &= \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=2} \Gamma(s)(2\pi)^{-s} D(s) y^{-s} ds = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=2} \Lambda(s) y^{-s} ds \end{aligned}$$

Now use the functional equation $\Lambda(s) = i^k \Lambda(k-s)$, change variables, shift contours and eventually recover $(iy)^{-k} f(i/y)$. Along the way one needs to use that $D(s)$ is bounded in vertical strips.